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Irreducible $*$-Representations of Some Group Rings and Associated Banach $*$-Algebras*

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Let G be the group which is a wreath product of two infinite cyclic groups. We construct a faithful, topologically irreducible $*$-representation of $l^1(G)$ on $l^2(\mathbb{Z})$, and a faithful, strictly irreducible representation on $l^1(\mathbb{Z})$. The finite-dimensional, irreducible $*$-representations of $l^1(G)$ are described, and we find that an associated C^*-algebra is primitive and antiliminal, with a separating set of finite-dimensional, irreducible representations. A similar study is made of the group H generated by x and y with the relation $x^{-1}yx = x^2$.

1. INTRODUCTION

Given a group G, let $k[G]$ denote the ordinary group ring over a field k and $l^1(G)$ the Banach algebra consisting of elements

$$a = \sum a_g g,$$

with $a_g \in \mathbb{C}$ and $\|a\| = \sum |a_g| < \infty$. Both $\mathbb{C}[G]$ and $l^1(G)$ have an involution $*$ given by

$$\left(\sum a_g g\right)^* = \sum \bar{a}_g g^{-1},$$

with respect to which $\mathbb{C}[G]$ is a dense $*$-subalgebra of $l^1(G)$. In case G is a non-abelian free group of countable rank, it follows as a special case of a theorem of Formanek that $\mathbb{C}[G]$ is primitive [2], and McGregor has extended this by constructing faithful, strictly irreducible $*$-representations of $\mathbb{C}[G]$ and $l^1(G)$ [9].

In Sections 2 and 3, we consider the group $\mathbb{Z} \wr \mathbb{Z}$. We have proved in [5]

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that $k[\mathbb{Z} \wr \mathbb{Z}]$, and more generally $k[\mathbb{Z} \wr K]$, is primitive for any field k and infinite group K. In Section 2, we construct a faithful, topologically irreducible $*$-representation of $l^1(\mathbb{Z} \wr \mathbb{Z})$ on the Hilbert space $l^2(\mathbb{Z})$, and a faithful, strictly irreducible representation on $l^1(\mathbb{Z})$. We also show that with regard to the representation on $l^2(\mathbb{Z})$, the closure C of the image of $l^1(\mathbb{Z} \wr \mathbb{Z})$ contains no non-zero compact operator, from which it follows that C is an antiliminal (NGCR) C^*-algebra. The constructions of this section may be extended to produce faithful, topologically irreducible $*$-representations of $l^1(\mathbb{Z} \wr K)$ on $l^2(K)$ and faithful, strictly irreducible representations on $l^1(K)$. In particular, $l^1(\mathbb{Z} \wr K)$ is primitive.

We have determined the finite-dimensional, irreducible representations of $k[\mathbb{Z} \wr \mathbb{Z}]$ in [7]. In Section 3, we use these results to list the finite-dimensional, irreducible $*$-representations of $l^1(\mathbb{Z} \wr \mathbb{Z})$ and C, and find that there are enough such representations to separate all the elements of C.

Guichardet first provided an example which, like C, is a primitive, antiliminal C^*-algebra with a separating set of finite-dimensional, irreducible representations. His example is the enveloping C^*-algebra of the group ring $\mathbb{C}[H]$, where H is the group generated by x and y with the relation $x^{-1}yx = y^2$. In Section 4, we construct a faithful, irreducible $*$-representation of $\mathbb{C}[H]$ on $l^2(\mathbb{Z})$, and show that its norm closure is also primitive and antiliminal, with a separating set of finite-dimensional, irreducible representations.

2. The Faithful Representation of $l^1(\mathbb{Z} \wr \mathbb{Z})$

Let us write G for the group $\mathbb{Z} \wr \mathbb{Z}$, which we present with generators $\{y_n : n \in \mathbb{Z}\} \cup \{x\}$ and the relations

$$y_m y_n = y_n y_m, \qquad x^{-1} y_n x = y_{n+1}.$$

Let F denote the free abelian group generated by the y's. A typical element of $l^1(G)$ has the form

$$\sum_{i,m} c(i, m)\, m x^i,$$

with m a word in F, the exponent i an integer, and $\{c(i, m)\}$ a sequence of complex numbers such that $\sum |c(i, m)| < \infty$.

THEOREM 2.1. *The algebra $l^1(G)$ has a faithful, topologically irreducible $*$-representation into the algebra B of bounded linear operators on $l^2(\mathbb{Z})$.*

Proof. Let $\{v_n : n \in \mathbb{Z}\}$ be an orthonormal basis for $l^2(\mathbb{Z})$. We first describe the action of G as a group of unitary operators on $l^2(\mathbb{Z})$. Let

$$x \cdot v_n = v_{n+1},$$
$$y_0 \cdot v_n = c_n v_n,$$

where c_n is a complex number of absolute value 1. We choose the sequence $\{c_n\}$ as follows: let S be a countable collection of complex numbers of absolute value 1 including all the roots of unity. Let the complex numbers c_n be chosen from S so that any possible finite sequence in S occurs infinitely often as a subsequence $c_n, c_{n+1}, \ldots, c_{n+r}$.

This defines the action of the other elements y_m, since $y_m = x^{-m} y_0 x^m$, and we find that

$$y_m \cdot v_n = c_{m+n} v_n.$$

The choice of the scalars c_m on the circle ensures that each element of G acts as a unitary operator. The action extends linearly to an action of $l^1(G)$ on $l^2(\mathbb{Z})$, and the resulting representation of $l^1(G)$ into B is evidently a norm-decreasing $*$-representation.

To prove faithfulness, it suffices to show that $l^1(F)$ acts faithfully. For any element of $l^1(G)$ can be written as $\sum p_i x^i$ with each p_i in $l^1(F)$, and if this annihilates $l^2(\mathbb{Z})$, then

$$0 = \left(\sum p_i x^i\right) \cdot v_n = \sum p_i \cdot v_{i+n}$$

for all n. Thus each p_i annihilates all the v_{i+n}.

Let $p = \sum_{i=0}^{\infty} a_i m_i(\mathbf{y}_k)$ be a non-zero element of $l^1(F)$, with $a_i \in \mathbb{C}$ and $m_i \in F$. Then p acts diagonally on the basis of $l^2(\mathbb{Z})$, with

$$p \cdot v_n = \sum_{i=0}^{\infty} a_i m_i(\mathbf{c}_{k+n}) \cdot v_n.$$

We will assume that p annihilates $l^2(\mathbb{Z})$, and obtain a contradiction. Multiplying p by a word in F if necessary, we may assume that p has a non-zero constant term a_0. Choose an N so that $\sum_{i>N} |a_i| < |a_0|$ and suppose (conjugating p by a power of x if necessary) that the monomials $m_0, \ldots, m_N$ involve only the variables $y_0, \ldots, y_{t-1}$ and their inverses. Let l be a large prime number and consider the element

$$q = \sum_{i=0}^{l-1} x^{-ti} p x^{ti} = \sum_{j=0}^{\infty} a_j \sum_{i=0}^{l-1} m_j(\mathbf{y}_{k+ti}).$$

Write $\bar{m}_j$ for the polynomial $\sum_{i=0}^{l-1} x^{-ti} m_j x^{ti}$, so that $q = \sum_{j=0}^{\infty} a_j \bar{m}_j$.

The polynomial q annihilates $l^2(\mathbb{Z})$, so that for any ordered substitution of the c's for the y's, the sum is 0. Since all the c's have absolute value 1, we see that under any substitution,

$$\left| \sum_{j>N} a_j \bar{m}_j(\mathbf{c}) \right| \leqslant l \sum_{j>N} |a_j| < l\,|a_0|.$$

Thus we will have a contradiction if we show that there is a substitution for which

$$\left| \sum_{j=0}^{N} a_j \bar{m}_j(\mathbf{c}) \right| = l\,|a_0|.$$

The monomial m_0 is assumed to be 1, so that $\bar{m}_0 = l$, and it suffices to find a substitution $c_{k+1},\dots, c_{k+tl}$ for which each of $\bar{m}_1,\dots, \bar{m}_N$ vanishes. We have

$$\begin{aligned}\bar{m}_j(c_{k+1},\dots, c_{k+tl}) = m_j(c_k,\dots, c_{k+t}) + m_j(c_{k+t+1},\dots, c_{k+2t}) + \cdots \\ + m_j(c_{k+(l-1)t+1},\dots, c_{k+tl}).\end{aligned} \tag{1}$$

Suppose $c_{k+1},\dots, c_{k+t}$ are lth-roots of unity such that each of $m_1(c_{k+1},\dots, c_{k+t}),\dots, m_N(c_{k+1},\dots, c_{k+t})$ is not equal to 1. Taking l to be as large a prime number as necessary, we can find such a sequence of roots of unity. Suppose further that for $0 < i < l$ and $0 < j \leqslant t$, we have

$$c_{k+it+j} = (c_{k+j})^{i+1},$$

so that the ith-summand on the right side of (1) is the $(i+1)$st-power of the first summand. Then each $\bar{m}_j(c_{k+1},\dots, c_{k+tl})$ is a sum of l distinct lth-roots of unity, which is 0. Since the sequence $\{c_m\}$ includes as subsequences every possible finite sequence of roots of unity, we can choose an integer k for which $c_{k+1},\dots, c_{k+tl}$ are as above, obtaining the desired contradiction.

It remains to prove that the representation is topologically irreducible, or that every vector is topologically cyclic. For any positive integer t, there is a sequence of c's of length $2t+1$ consisting only of 1's, and another sequence of length $2t+1$ consisting of $2t$ many 1's, with a -1 in the middle. Let the index of the central term in each sequence be i and j, respectively, and let $z_t = \frac{1}{2}(y_i - y_j)$. Then

$$\begin{aligned} z_t \cdot v_0 &= v_0, \\ z_t \cdot v_n &= 0 \qquad \text{for} \quad 0 < |n| \leqslant t, \end{aligned}$$

and for $|n| > t$, the operator z_t sends v_n to a multiple by a scalar of absolute value $\leqslant 1$. Thus, given an arbitrary $v = \sum a_i v_i$ in $l^2(\mathbb{Z})$ with $a_0 \neq 0$ and $\varepsilon > 0$, we may choose t so that

$$z_t \cdot v = a_0 v_0 + w$$

with $\|w\| < \varepsilon$. For any other element $\sum_j b_j v_j$ in $l^2(\mathbb{Z})$, we have

$$\sum_{j=-n}^{n} (b_j/a_0)\, x^j z_t v = \sum_{j=-n}^{n} b_j v_j + \left(\sum_{j=-n}^{n} (b_j/a_0)\, x^j \right) w.$$

Thus,

$$\left\| \sum_j b_j v_j - \sum_{j=-n}^{n} (b_j/a_0)\, x^j z_t v \right\|$$
$$\leqslant \left(\sum_{|j|>n} |b_j|^2 \right)^{1/2} + \left(\sum_{j=-n}^{n} |b_j/a_0| \right) \|w\|. \tag{2}$$

By choosing ε sufficiently small and n sufficiently large, we may make this quantity arbitrarily small, proving that v is topologically cyclic. ∎

Remark. By using a similar argument, one can construct for any infinite group K a faithful, topological irreducible $*$-representation of $l^1(\mathbb{Z} \wr K)$ on $l^2(K)$. The construction is slightly more technical, since scalars $\{c_k : k \in K\}$ must be defined inductively, rather than chosen in advance as above. The basic idea is contained in the proof that the group ring of $\mathbb{Z} \wr K$ is primitive in [5].

THEOREM 2.2. *The algebra $l^1(G)$ has a faithful, strictly irreducible representation on $l^1(\mathbb{Z})$ which is norm-decreasing. In particular, $l^1(G)$ is primitive.*

Proof. Let $\{v_n : n \in \mathbb{Z}\}$ be a basis for $l^1(\mathbb{Z})$, and let the action of G on this basis be defined exactly as in the proof of Theorem 2.1. Each element of G acts as an operator of norm 1, from which it follows that we can extend the action to a norm-decreasing representation of $l^1(G)$. We may follow the proof of Theorem 2.1 word for word to show that the representation is faithful and topologically irreducible, except for a slight change in (2).

The new ingredient in this case is that each basis vector v_n is strictly cyclic. For, given $v = \sum a_i v_i$ in $l^1(\mathbb{Z})$, we have

$$\sum_i a_i x^{i-n} \quad v_n = v,$$

and $\sum_i a_i x^i \in l^1(G)$. It follows from topological irreducibility and the existence of a strictly cyclic vector that the representation is strictly irreducible [8, p. 159].

Remark. This result also extends to $l^1(\mathbb{Z} \wr K)$ for any infinite group K; that is, $l^1(\mathbb{Z} \wr \mathbb{K})$ is primitive, with a faithful, strictly irreducible representation on $l^1(K)$.

We return to the setup of Theorem 2.1.

THEOREM 2.3. *The C^*-algebra C obtained as the norm closure of the image of $l^1(G)$ in B contains no compact operator besides* 0. *Thus C is antiliminal.*

Proof. The algebra C acts irreducibly on $l^2(\mathbb{Z})$, so that if it contains one non-zero compact operator, it contains them all [1, 4.1.10]. Let T be the compact operator defined by

$$T \cdot v_0 = v_0,$$
$$T \cdot v_n = 0 \qquad \text{for} \quad n \neq 0.$$

We will show that the ball of radius $\frac{1}{2}$ around T contains no element of the group ring $\mathbb{C}[G]$. Since this is dense in C, it follows that T cannot lie in C.

Any element of $\mathbb{C}[G]$ has the form

$$p = \sum_{i=-m}^{m} p_i(\mathbf{y}_k)\, x^i.$$

Applying $p - T$ to v_0 yields

$$(p_0(\mathbf{c}_k) - 1)\, v_0 + \sum_{0 < |i| \leqslant m} p_i(\mathbf{c}_{k+i})\, v_i,$$

so that $\|p - T\| \geqslant |p_0(\mathbf{c}_k) - 1|$. On the other hand, for infinitely many n, the sequence of scalars $c_k, c_{k+1}, ..., c_l$ which is substituted in p_0 above recurs as $c_{k+n}, ..., c_{l+n}$. Thus we can find an n such that

$$(p - T)\, v_n = p_0(c_{k+n}, ..., c_{l+n}) \cdot v_n + (\text{terms not involving } v_n)$$

and

$$\|p - T\| \geqslant |p_0(c_{k+n}, ..., c_{l+n})| = |p_0(\mathbf{c}_k)|.$$

Suppose $\|p - T\| < \frac{1}{2}$. Then the two numbers $|p_0(\mathbf{c}_k)|$ and $|p_0(\mathbf{c}_k) - 1|$ are $< \frac{1}{2}$, a contradiction. This proves that C cannot contain T, or any other compact operator besides 0.

By [1, 4.2.6], we may conclude that (0) is the largest liminal closed two-sided ideal of C, so that C is antiliminal by definition.

3. Finite-Dimensional Representations

We have classified the finite-dimensional, irreducible representations of $k[G]$ for any field k in [7]. We recall that those of degree n arise as representations of the twisted Laurent polynomial extension A_n of $S = k[y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}]$ obtained by adjoining variables x and x^{-1} with the rule

$$x^{-1} y_i x = y_{i+1} \qquad \text{for} \quad i < n,$$
$$x^{-1} y_n x = y_1 .$$

The basic result is that for k algebraically closed, every irreducible representation of A_n of degree n is equivalent to one of the form

$$y_n \mapsto \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \ldots, y_n \mapsto \begin{pmatrix} a_n & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_{n-1} \end{pmatrix},$$

$$x \mapsto \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ b & & & 0 \end{pmatrix},$$

where $a_i, b \in k^*$ and the orbit of $(a_1, \ldots, a_n)$ under cyclic permutation of the indices has size n. The resulting space of maximal ideals $\operatorname{Spec}_n A_n$ can be explicitly described. Let $R = S^x$ be the fixed algebra with respect to the automorphism given by conjugation by x. The center of A_n is $R[x^n, x^{-n}]$, and $\operatorname{Spec}_n A_n$ is homeomorphic to the open subscheme of $\operatorname{Spec} R[x^n, x^{-n}]$ which is the complement of the branch locus with respect to the cover by $\operatorname{Spec} S[x^n, x^{-n}]$. Using these results, we obtain

Proposition 3.1. *The irreducible $*$-representations of $\mathbb{C}[G]$ or $l^1(G)$ on an n-dimensional Hilbert space are equivalent to representations of the form*

$$y_1 \mapsto \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ b & & & 0 \end{pmatrix},$$

where a_i, b are complex number of absolute value 1 *and $(a_1, \ldots, a_n)$ has an orbit of size n with respect to cyclic permutation of the indices.*

Proof. In order for the representations of $\mathbb{C}[G]$ to respect the involution, y_1 and x must map to unitary matrices. Thus the above are the only possibilities for $\mathbb{C}[G]$. Since $\mathbb{C}[G]$ is dense in $l^1(G)$, the result holds for $l^1(G)$ as well.

In Section 2, we embedded $l^1(G)$ into the algebra B of bounded linear operators on $l^2(\mathbb{Z})$. The map is a $*$-algebra isomorphism onto its image, but may be norm decreasing. We defined C to be the norm closure of the image in B. Thus $\mathbb{C}[G]$ and $l^1(G)$ map to isomorphic $*$-algebras which are dense in C. It follows that the finite-dimensional, irreducible $*$-representations of C are precisely those of $\mathbb{C}[G]$ and $l^1(G)$, namely, those given in Proposition 3.1. The space $\operatorname{Spec}_n C$ can then be described as a closed subset of $\operatorname{Spec}_n \mathbb{C}[G]$, corresponding to the closed set in $\operatorname{Spec} S[x^n, x^{-n}]$ with coordinates of absolute value 1. Thus, given the quotient of the $(n+1)$-torus with respect to cyclic permutation of the first n coordinates, $\operatorname{Spec}_n C$ is the complement of the resulting branch locus. We can now prove the result we want for C:

THEOREM 3.2. *The algebra C has enough finite-dimensional, irreducible representations to separate all of its elements.*

LEMMA 3.3. *Given an element q in $\mathbb{C}[G]$, and $\varepsilon > 0$, there is a finite-dimensional, irreducible $*$-representation ψ of $\mathbb{C}[G]$ such that $\|\psi(q)\| > \|q\|(1-\varepsilon)$, where the norm on $\mathbb{C}[G]$ is that inherited from C.*

Proof of Theorem 3.2. Let $p \neq 0$ be an element of C with $\|p\| = \alpha$ and choose q in $\mathbb{C}[G]$ with $\|p-q\| < \alpha/2$. Let ψ be the representation of $\mathbb{C}[G]$ given by Lemma 3.3. Then ψ extends to an irreducible representation of C. If $\psi(p) = 0$, then

$$\|p-q\| \geqslant \|\psi(p)-\psi(q)\| = \|\psi(q)\| > \|q\|(1-\varepsilon).$$

But $\|q\| \geqslant \|p\| - \|p-q\| > \alpha/2$, so that we obtain

$$\|p-q\| > (1-\varepsilon)\,\alpha/2.$$

Since ε may be arbitrarily small, this implies that $\|p-q\| \geqslant \alpha/2$, a contradiction.

Proof of Lemma 3.3. Choose $v \in l^2(\mathbb{Z})$ with $\|v\| = 1$ and $\|q \cdot v\| \geqslant \|q\|(1-\varepsilon/2)$. Write $v = \sum a_i v_i$ and choose $m > 0$ so that

$$\left\| \sum_{|i|>m} a_i v_i \right\| < \varepsilon/2.$$

Then

$$\|q \cdot v\| \leqslant \left\| q\left(\sum_{i=-m}^{m} a_i v_i \right) \right\| + \left\| q \sum_{|i|>m} a_i v_i \right\|$$

so that

$$\left\| q\left(\sum_{i=-m}^{m} a_i v_i\right)\right\| \geqslant \|q \quad v\| - \|q\|\,\varepsilon/2 \geqslant \|q\|\,(1-\varepsilon).$$

We may choose an integer n so that the sequence $c_{-n},\dots, c_n$ of Section 2 has an orbit of size $2n+1$ with respect to cyclic permutation of the indices and so that $n-m$ is greater than the absolute value of any exponent of x in q and any index of y in q. Let ψ be the $(2n+1)$-dimensional representation of $\mathbb{C}[G]$ on a space with basis $w_{-n},\dots, w_n$, given by

$$\begin{aligned} \psi(y_0)\cdot w_i &= c_i w_i, \\ \psi(x)\cdot w_i &= w_{i+1} \qquad \text{for} \quad i<n, \\ \psi(x)\cdot w_n &= w_{-n}. \end{aligned}$$

Then ψ is an irreducible $*$-representation, and if

$$q\left(\sum_{i=-m}^{m} a_i v_i\right) = \sum b_i v_i,$$

then

$$\psi(q)\cdot\left(\sum_{i=-m}^{m} a_i w_i\right) = \sum b_i w_i.$$

Thus $\|\psi(q)\| \geqslant \|q\|\,(1-\varepsilon)$, and the theorem is proved.

4. The Example of Guichardet

Let H be the group generated by x and y with the relation $x^{-1}yx = y^2$. It is also convenient to present H in a different manner. Using the presentation of $\mathbb{Z} \wr \mathbb{Z}$ given in Section 2, then H is isomorphic to the image of $\mathbb{Z} \wr \mathbb{Z}$ obtained by letting $y_n^2 = y_{n+1}$ for all n. Thus we may think of H as generated by elements $x, y, y^{1/2}, y^{1/4},\dots$ with the relations

$$x^{-1} y^{1/2^n} x = y^{1/2^{n-1}}.$$

Guichardet has studied the enveloping C^*-algebra of $\mathbb{C}[H]$ in considerable detail [3]. We shall obtain some related results.

Theorem 4.1. *The group ring $\mathbb{C}[H]$ has a faithful, topologically irreducible $*$-representation on $l^2(\mathbb{Z})$, and there is no compact operator in the closure D of its image, besides* 0.

Proof. Let $\{v_n : n \in \mathbb{Z}\}$ be an orthonormal basis for $l^2(\mathbb{Z})$ as before. Let $\{c_n : n \in \mathbb{Z}\}$ be a sequence of roots of unity with the property that

$$c_n^2 = c_{n+1}, \qquad c_0 = 1,$$

and for $n < 0$ assume that $c_n \neq 1$. To define the representation, let

$$x \cdot v_n = v_{n+1},$$
$$y \cdot v_n = c_n v_n.$$

The choice of scalars ensures that the relation $x^{-1}yx = y^2$ is satisfied, and that each group element acts as a unitary operator. It follows that this extends to a $*$-representation of $\mathbb{C}[H]$.

To prove faithfulness, it suffices to show that no non-zero element of $\mathbb{C}[P]$ annihilates $l^2[\mathbb{Z}]$, where P is the subgroup generated by $\{y, y^{1/2},...\}$. By a change of variable, we may regard any such element as a polynomial p in y and y^{-1}. If $p \quad v_n = 0$ for all n, then $p(c_n) = 0$ for all n. Thus p vanishes on an infinite set, and must be the 0 polynomial.

For topological irreducibility, we must show that every non-zero vector is topologically cyclic. Let $z = (1/2i)\, x^2(y - y^{-1})\, x^{-2}$. Then

$$z \cdot v_n = \gamma_{n-2} v_n,$$

where γ_n is the imaginary part of c_n. Assuming that c_{-2} was chosen to equal i, we find that

$$\begin{aligned} \gamma_n &= 0 \qquad \text{for} \quad n > -2, \\ \gamma_{-2} &= 1, \\ |\gamma_n| &< 1 \qquad \text{for} \quad n < -2. \end{aligned}$$

Given an arbitrary vector $v = \sum a_i v_i$ with $a_0 \neq 0$, and any $\varepsilon > 0$, we may choose t so that

$$z^t \quad v = a_0 v_0 + w$$

with $\|w\| < \varepsilon$. It follows easily, as in Theorem 2.1, that v is topologically cyclic.

Finally, let T be the projection operator defined by $Tv_0 = v_0$ and $Tv_n = 0$ for $n \neq 0$. Then, as in Theorem 2.3, the open ball of radius 1/2 around T contains no element of $\mathbb{C}[G]$. For, given

$$p = \sum_{i=-m}^{m} p_i(y, y^{-1})\, x^i$$

(changing variables if necessary so that each p_i is a polynomial in y, y^{-1}), we have

$$(p - T) \cdot v_0 = (p_0(1) - 1)\, v_0 + \sum_{0<|i|\leqslant m} p_i(c_i)\, v_i,$$

so that $\|p - T\| \geqslant |p_0(1) - 1|$. But for any $n > 0$, we have

$$(p - T) \cdot v_n = p_0(1) \cdot v_n + \sum_{0<|i|\leqslant m} p_i(c_{n+i})\, v_{n+i},$$

implying that $\|p - T\| \geqslant |p_0(1)|$. It follows that $\|p - T\| \geqslant 1/2$, so that D does not contain the compact operator T.

Remark. Suppose we let $\mathbb{C}[H]$ act on the vector space V of countable dimension with basis $\{v_n : n \in \mathbb{Z}\}$ exactly as above. The same proof shows that V is faithful, and it is easy to show that V is strictly irreducible as well. Thus $\mathbb{C}[H]$ is primitive. The same idea applies to prove primitivity of $k[H]$ for any field k which is not algebraic over the field of two elements. Alternatively, a more complicated faithful, simple module can be constructed for $k[H]$ in case k has characteristic 2. Of course, this extends to groups generated by x and y with the relation $x^{-1}yx = y^r$, for $r > 1$. See [6] for these results.

The preceding theorem implies that D is an antiliminal C^*-algebra. We will classify the finite-dimensional, irreducible representations of D by computing those for $\mathbb{C}[H]$. Given any field k, we can form the Ore extension E of $k[y, y^{-1}]$ by adjoining the variable x with respect to the relations $yx = xy^2$ and $y^{-1}x = xy^{-2}$. The finite-dimensional, irreducible representations of E were determined in [4, 8.4], and in every such representation which is not of degree one, x is sent to an invertible matrix. Thus these are precisely the representations of $k[H]$:

THEOREM 4.2. *Let k be an algebraically closed field. The irreducible representations of $k[H]$ of dimension > 1 are equivalent to representations of the form*

$$y \mapsto \begin{pmatrix} a & & & \\ & a^2 & & \\ & & \ddots & \\ & & & a^{2^n} \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ b & & & 0 \end{pmatrix}, \tag{3}$$

with $b \neq 0$ and $a^{2^n} = a$, but $a^{2^i} \neq a$ for $i < n$. The kernel of this representation is

$$(x^n - b, (y - a) \cdots (y - a^{2^{n-1}})).$$

Proof. This is a special case of [4, 8.4].

COROLLARY 4.3. *The irreducible $*$-representations of* $\mathbb{C}[H]$ *on a Hilbert space of dimension n are equivalent to representations of the form given in* (3), *with* $|b| = 1$.

Proof. The image of y is unitary, since a is a root of unity, and the image of x is unitary precisely if $|b| = 1$.

Note that in this case the space $\mathrm{Spec}_n\, k[H]$ is one-dimensional for any n, being the union of a finite number of copies of k^*. Correspondingly, the space $\mathrm{Spec}_n\, D$ is a finite number of copies of the circle. Our final result is that these representations form a separating set for D.

THEOREM 4.4. *The algebra D has enough finite-dimensional, irreducible representations to separate all of its elements.*

Proof. As in the proof of Theorem 3.2, it suffices to prove an analogue of Lemma 3.3: for any element q of $\mathbb{C}[H]$ and $\varepsilon > 0$ there is a finite-dimensional, irreducible $*$-representation ψ of $\mathbb{C}[H]$ for which $\|\psi(q)\| > \|q\|(1-\varepsilon)$, where the norm on $\mathbb{C}[H]$ is that inherited from D. Unlike in the proof of Lemma 3.3, we can no longer find for any n a finite-dimensional, irreducible representation ψ of $\mathbb{C}[H]$ in which the eigenvalues of $\psi(y)$ are $c_{-n},..., c_n$. Nevertheless, we can get as close as we desire, and this is sufficient to carry out the argument, with slight changes.

Specifically, for n fixed, the equality

$$\left(\frac{1}{2^n} - \frac{2^m}{2^{n+m}-1}\right) = \frac{-1}{2^n(2^{n+m}-1)},$$

shows that for m large enough, the difference between the two left-hand fractions is arbitrarily small. Therefore, we can find a primitive $(2^{n+m}-1)$th-root of unity c arbitrarily close to c_{-n}. The irreducible representation ψ of dimension $n+m$ given by (3), with c chosen as above, closely approximates the action of $\mathbb{C}[H]$ on the subspace of $l^2(\mathbb{Z})$ spanned by $v_{-n},..., v_n$, provided m is large enough. This allows us to mimic the proof of Lemma 3.3; we omit the details.

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REFERENCES

1. J. DIXMIER, "C^*-algebras," North-Holland, Amsterdam, 1977. (This is a translation of "Les C^*-algèbres et leurs représentations," Gauthier–Villars, Paris, 1969.)

2. E. FORMANEK, Group rings of free products are primitive, *J. Algebra* **26** (1973), 508–511.
3. A. GUICHARDET, Caractères des algèbres de Banach involutives, *Ann. Inst. Fourier (Grenoble)* **13** (1963), 1–81.
4. R. IRVING, Prime ideals of Ore extensions over commutative rings, *J. Algebra* **56** (1979), 315–342.
5. R. IRVING, Some primitive group rings, *J. Algebra* **56** (1979), 274–281.
6. R. IRVING, Some more primitive group rings, *Israel J. Math.*, in press.
7. R. IRVING, Finite-dimensional, irreducible representations of some crossed products and group rings, *J. Pure Appl. Algebra*, in press.
8. C. MCGREGOR, A representation for $l^1(S)$, *Bull. London Math. Soc.* **8** (1976), 156–160.
9. C. MCGREGOR, On the primitivity of the group ring of a free group, *Bull. London Math. Soc.* **8** (1976), 294–298.